

A New Kind of McKay Correspondence From Non-Abelian Gauge Theories

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Abstract

The boundary chiral ring of a 2d gauged linear sigma model on a Kähler manifold X classifies the topological D-brane sectors and the massless open strings between them. While it is determined at small volume by simple group theory, its continuation to generic volume provides highly non-trivial information about the D -branes on X , related to the derived category $D^b(X)$. We use this correspondence to elaborate on an extended notion of McKay correspondence that captures more general than orbifold singularities. As an illustration, we work out this new notion of McKay correspondence for a class of non-compact Calabi-Yau singularities related to Grassmannians.

March 2001

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1. Introduction

The understanding of D-brane dynamics on curved spaces with background fields is an important question. One aspect of great interest is to connect the different perturbative descriptions of the D-branes in different regimes of the moduli space². A simple description of D-branes arises on manifolds X with small curvatures, where they may be interpreted in terms of equivalence classes of vector bundles, or more generally, of coherent sheaves on X . In this region of the moduli space the appropriate description is in terms of the K-theory group $K(X)$ [3], or for a more refined description, the derived category $\mathcal{D}^b(X)$ [2]. By a variation of the Kähler volumes of X one may then interpolate to small volume, or large curvatures, where the geometric picture is corrected by world-sheet quantum effects. Here the appropriate description is in terms of the two-dimensional conformal theory on the world-sheet of the string. Although the corrections may be large, one nevertheless expects a close correspondence between the D-brane objects at small and large volumes, due to the decoupling hypothesis [4].³ It states that the holomorphic objects do not depend on the Kähler volume and thus the string corrections enter only at the level of stability questions of the holomorphic branes.

It was argued in [6] that the holomorphic objects may be classified by the boundary chiral ring \mathcal{R}_Δ in a two-dimensional gauge theory with vacuum geometry X . Indeed the large volume continuation of the holomorphic small volume objects generated by \mathcal{R}_Δ turn out to have some rather miraculous properties, as will be reviewed below. For example, there is a canonical identification of certain truncated modules of \mathcal{R}_Δ with free generators for $\mathcal{D}^b(X)$. A remarkable aspect is that the geometry in question is the quantum corrected version, which means that this approach allows to bypass the usual – quite sophisticated – methods of mirror symmetry with simple algebraic techniques pertaining to the boundary chiral ring.

The interpolation to large volume provides an amazingly simple way to extract highly non-trivial geometrical structures from the simple group theory of the boundary ring. This leads naturally to a generalized notion of “McKay correspondence”⁴ as the isomorphism (predicted by the phase picture of the 2d gauge theory) between the representations of the 2d boundary ring and their large volume sheaf duals.

² See [1] for an introduction and [2] for a more recent list of references.

³ See also [5].

⁴ We refer to [7,8] for an overview and references on the classical McKay correspondence.

These ideas apply to any dimension and to geometries involving algebraic constraints, and lead to a construction of such a correspondence for any, possibly non-unique or only partial, resolution. In particular the notion of a discrete group, central to the original McKay correspondence, is replaced by the continuous gauge group H ; e.g. the intersections of the compact homology, which coincide with the topological open string index, are determined by the structure constants of the boundary ring, which is isomorphic to a truncation of the H representation ring [6]. Whether or not it is possible (and desirable at all) to reduce H to a discrete group, as for the classical McKay correspondence, is of little relevance in this context.

The purpose of this note is to substantiate and test these ideas in the more general case where the gauge group is non-Abelian, $H = U(k)$, and the exceptional divisor of the resolution $X \rightarrow \hat{X}$ is a Grassmannian $G_{k,n}$. The study of these geometries for general k is interesting because their small-volume structure is not an orbifold, in contrast to what the traditional notion of McKay correspondence is based upon. Rather, they correspond to more general quotients, for which the rôle of the discrete orbifold group is played by a certain subgroup Γ' of the continuous gauge group $U(k)$. In particular the “tautological sheaves” are no longer line bundles as in previous cases. These geometries thus provide a good testing ground for the advertised definition of a generalized McKay correspondence in terms of gauged linear sigma models, while allowing for comparison with independent results of mathematicians (most notably Kapranov’s [9]⁵).

The organization of this note is as follows. In sect. 2 we give a brief summary of the results of [6], in particular how the boundary ring of a certain 2d gauge theory “generates” the topological sector \mathcal{H}_{op}^{top} of the open string Hilbert space⁶. In sect. 3 we determine the boundary ring \mathcal{R}_Δ for a special class of gauge theories with gauge group $U(k)$. The small volume vacuum geometry is a (non-compact) Calabi–Yau quotient singularity \hat{X} obtained in the zero size limit of a Grassmannian hypersurface $G_{k,n}$, embedded as an exceptional divisor in the resolution $X \rightarrow \hat{X}$. We construct two bases $\{R_a\}$ and $\{S^a\}$ for the topological open string Hilbert space \mathcal{H}_{op}^{top} from \mathcal{R}_Δ , and determine the index in terms of a truncation of the representation ring of H . In sect. 4 we interpolate to large volume (small curvature) and identify the large volume images $\{R_a^\infty\}$ and $\{S_\infty^a\}$ of the holomorphic small volume bases as collections of exceptional sheaves that generate freely the derived category $\mathcal{D}^b(X)$. Specifically, $\{S_\infty^a\}$ represents

⁵ That one should be able to derive Kapranov’s results via a generalized McKay correspondence has been conjectured by M. Reid.

⁶ It seems quite possible that this correspondence may be generalized to the non-topological sector as well.

the homology $H_{*c}(X)$ with compact support and the group theoretical topological index coincides with the intersection form on the latter. In sect. 5 we illustrate these ideas in a case study, for which we work out the fractional brane content, the quiver diagram and an alternative description in terms of a local mirror LG model. In sect. 6 we conclude with some comments on generalizations. In particular we study D-branes on complete intersection Calabi-Yau 3-folds in Grassmannians.

2. Boundary chiral rings and the derived category $\mathcal{D}^b(X)$

We start with a review of the correspondence between the boundary chiral ring \mathcal{R}_Δ and the topological sector of the open string Hilbert space and its relation to the McKay correspondence [6]. The general properties of the boundary chiral ring and its relation to the derived category will be explored in more depth in [10]; here we restrict to a formulation and explanation of the proposal and outline some general arguments. In brief, the proposal says that the most basic objects of the topological open string sector on a Kähler manifold X , namely the allowed boundary conditions and their massless open string spectra, are classified by the multiplication ring of chiral fields of a 2d gauge theory at the boundary⁷. As the topological data are closely related to the derived category $\mathcal{D}^b(X)$, this provides also the link between group theory and geometry that leads to a significant extension of the notion of “McKay correspondence”.

2.1. The boundary chiral ring

As is well-known, the zero mode sector of the closed string Hilbert space \mathcal{H}_{cl}^{top} on a Kähler manifold X has a topological structure. It is described by a deformation of the cohomology ring $H^{k,k}(X)$, called the quantum cohomology ring. The latter is isomorphic to the quantum chiral ring \mathcal{R}_{cl} [11], which is the ring of chiral primary field operators in the $(2, 2)$ super-conformal 2d world-sheet theory of the string without boundaries.

The open string sector is described by world-sheets with spatial boundaries in the 2d CFT. Physics-wise these correspond to D-branes in the string theory on X and their world-volume fields support coherent sheaves on submanifolds of X . The zero mode sector of the Hilbert space \mathcal{H}_{op}^{top} in the boundary sector has again a topological

⁷ As will be explained in [10], this follows from an isomorphism between the boundary fusion ring and the path algebra of a quiver construction, and the isomorphism between the derived categories of quiver algebras and of coherent sheaves. This gives an alternative derivation of the fact [2] that the category of topological D-branes is the derived category.

structure: it is isomorphic to the space of sections of $H^{0,k}(X, V)$ [12], where V is a coherent sheaf of the form $V = E_a^* \otimes E_b$ on X , where $E_{a,b}$ are the gauge bundles that couple to the two boundaries of the open strings, respectively.

Similar as for the closed string sector, the topological sector \mathcal{H}_{op}^{top} is isomorphic to a ring \mathcal{R}_Δ of 2d chiral field operators Ψ [6]. It may be conveniently described in terms of a gauged linear sigma model (GLSM), which is a $(2, 2)$ supersymmetric 2d gauge theory [13]. The ring \mathcal{R}_Δ is the multiplication ring of chiral matter fields of the GLSM projected to the boundary and it is accordingly referred to as the *boundary chiral ring*. In the IR, the GLSM model flows to a super-conformal fixed point and the ring \mathcal{R}_Δ flows to a boundary analog of the chiral bulk ring that describes the algebra of topological open string vertex operators. In complete analogy to the bulk chiral ring [11], the mutual OPE of the elements of the boundary ring is regular, since the dimensions of the fields are protected by their charges [10].

An important characteristic of the boundary chiral ring are its structure constants N which appear in the operator product:

$$(\Psi_\mu)_a^b (\Psi_\nu)_b^c \sim \sum_\rho N_{\mu\nu}{}^\rho(a, b, c) (\Psi_\rho)_a^c, \quad (2.1)$$

where $(\Psi_\mu)_a^b$ denotes a boundary field in the (a, b) sector labeled by μ . As far as fusions are concerned, the fields $(\Psi_\mu)_b^a$ can be replaced by semi-positive integral matrices $(A_\mu)_a^b$ which count the open string zero modes mapping between boundary conditions a and b . Then (2.1) turns into a topological analog of the familiar rational BCFT relation [14] between the annulus coefficients A and the Verlinde fusion matrices N (for this interpretation, we need to sum over b in (2.1)); however it is more general in that we do not need to require a rational CFT. An alternative, and quite interesting interpretation of (2.1) is as a matrix representation of the path algebra of the associated quiver theory.

Two essential novelties of the boundary chiral ring \mathcal{R}_Δ , as compared to the bulk chiral ring, are that: *i*) the ring elements carry non-trivial representations of the gauge group H , *ii*) there is a \mathbf{Z}_2 gradation corresponding to bosonic and fermionic chiral super-multiplets. The first property will be important for generating elements of $H^{0,p}(X, V)$ with non-trivial V . In fact, by a well-known property of the IR limit of the GLSM discussed in sect. 4, the vacuum bundle E for the H gauge fields is non-trivial and thus the boundary ring elements in non-trivial H representations become sections of a non-trivial bundle E .

The second property is a consequence of the partial supersymmetry breaking at the boundary; specifically the super-multiplets of the left-over supersymmetry at the boundary have bosonic and fermionic statistics in the directions tangential and normal

to the boundary, respectively [6][15]. This grading into bosonic and fermionic chiral fields at the boundary defines two sub-rings \mathcal{R}_Δ^+ and \mathcal{R}_Δ^- of \mathcal{R}_Δ , generated by the even and odd generators (not elements) of \mathcal{R}_Δ , respectively. This split gives rise to a construction of two generating bases for \mathcal{H}_{op}^{top} with remarkable properties. The claim is⁸ that acting with \mathcal{R}_Δ^\pm on a ground state $\mathcal{O}_X(m_\pm)$ of $U(1)$ charge m_\pm , one obtains naturally two finite bases $\{R_a\}$ and $\{S^a\}$ that generate freely the Hilbert space \mathcal{H}_{op}^{top} . As a first check of this claim one may consider the topological open string index for \mathcal{H}_{op}^{top}

$$\langle E_a, E_b \rangle = \text{Tr}_{ab}(-1)^F = \sum_k (-1)^k \dim \text{Ext}^k(E_a, E_b). \quad (2.2)$$

The r.h.s. is the natural expression for the index in the small volume phase, in that it is entirely determined by the group theoretical data (2.1) of the boundary chiral ring. In fact, modulo extra degeneracy factors from the global symmetry of the GLSM, the index is given by an alternating sum over the structure constants⁹ N of the fusion ring; these turn out to coincide with the structure constants of (a truncation of) the representation ring of H . From this one can directly verify that the intersection form on the bases $\{R_a\}$ and $\{S^a\}$ are invertible and that they span the lattice of RR charges of \mathcal{H}_{op}^{top} .

The language of the boundary chiral ring is particularly appropriate to describe the topological D-branes in a phase of the 2d gauge theory corresponding to “small volume” of X . In fact, for an appropriate choice of the GLSM, X is the resolution $X \rightarrow \hat{X}$ of a quotient singularity \mathbf{C}^M/Γ' , with $\Gamma' \subset H$ the (not necessarily discrete) quotient group. A change of parameters in the 2d gauge theory then interpolates from \hat{X} to the resolution X , by giving finite Kähler volume to the exceptional divisor $E \subset X$ [13].

Continuing the holomorphic objects in $\{R_a\}$ and $\{S^a\}$ to generic volume in this way leads to the following, remarkable link to the derived category $\mathcal{D}^b(X)$.

2.2. The derived category $\mathcal{D}^b(X)$ and McKay

Let $\{S_\infty^a\}$ and $\{R_\infty^a\}$ denote the generic, “large” volume counter parts of the bases $\{R_a\}$ and $\{S^a\}$ constructed from the boundary chiral ring at small volume. They are

⁸ There is a beautiful explanation of this fact that will be fully explored in [10].

⁹ In the present context the indices $\mu, \nu..$ and $a, b..$ run over the same set, moreover the matrices A coincide with the structure constants N (that is, they form the regular representation of the fusion algebra).

expected to be collections of coherent sheaves on X , and, by Hirzebruch-Riemann-Roch, the open string index has now a natural representation in terms of geometric integrals, i.e.:

$$(*) \quad \langle E_a, E_b \rangle = \sum_k (-1)^k \dim \operatorname{Ext}^k(E_a, E_b) \stackrel{HRR}{=} \int_X \operatorname{ch}(E_a^*) \operatorname{ch}(E_b) \operatorname{td}(X). \quad (2.3)$$

The large volume bases $\{S_\infty^a\}$ and $\{R_a^\infty\}$ enjoy some miraculous properties, summarized in the following conjecture [6]:

Conjecture: Let $\{R_a\}$ ($\{S^a\}$) denote a basis of $\chi(X)$ elements $\in \mathcal{H}_{op}^{top}$, obtained by acting with the sub-ring \mathcal{R}_Δ^+ of bosonic generators (\mathcal{R}_Δ^- of fermionic generators) of \mathcal{R}_Δ on a ground state $\mathcal{O}_X(m_+)$ ($\mathcal{O}_X(m_-)$). Then:

- i) The continuation of the bases $\{R_a^\infty\}$ and $\{S_\infty^a\}$ to large volume provides two bases of free generators for the derived category of coherent sheaves $\mathcal{D}^b(E)$. For an appropriate choice of a pair (m_+, m_-) of integers, they are orthogonal with respect to the inner product $(*)$.
- ii) The sheaves $\{R_a^\infty\}$ have a non-trivial extension to the non-compact space X and span $K(X)$. The sheaves $\{S_\infty^a\}$ have compact support on the exceptional divisor E and span the K-theory group $K_c(X)$ with compact support. The relation $(*)$ defines a ‘‘McKay correspondence’’ between the group theoretical data of a quotient group $\Gamma' \subset H$ and the intersections on the compact homology $H_{*c}(X)$.
- iii) The collections $\{R_a^\infty\}$ and $\{S_\infty^a\}$ are exceptional and generate helices \mathcal{H}_R and \mathcal{H}_S on E . The collection $\{S_\infty^a\}$ is a special mutation \underline{P} of $\{R_a^\infty\}$.

Note that the split into the K-theory groups with compact and non-compact support, which derives directly from the matter spectrum of the GLSM, is very much as in the formulation of the McKay correspondence by Ito and Nakajima [8]. Moreover the analytic continuation to large volume equates the group theoretical tensor products $\langle S^a, S^b \rangle = \text{l.h.s.}(2.3)$ of the small volume phase with the intersections $\langle S_\infty^a, S_\infty^b \rangle = \text{r.h.s.}(2.3)$ of the compact cohomology of the resolution, similarly as in the original McKay correspondence.

In this way the interpolation between small and large volume phases of boundary sector in the 2d gauge theory leads to a direct relation between the group theory data of a quotient singularity and the intersections of any (partial) resolution. This is in the spirit of the original McKay correspondence, and it agrees with the ideas of the mathematicians when restricted to those gauge theories that have at least two Higgs

phases, namely one that describes an orbifold singularity $\hat{X} = \mathbf{C}^n/\Gamma$ and another that describes a complete crepant resolution X of \hat{X} .

In the following we apply and test the above ideas for certain 2d gauge theories with gauge groups $H = U(k)$.

3. Boundary rings for $\mathcal{O}_G(-n)$ at small-volume

We will study now the GLSM with gauge group H with a vacuum geometry given by the non-compact Calabi–Yau $X = \mathcal{O}_G(-n)$, where $G = G_{k,n}$ is the Grassmannian parametrizing k -planes Λ_k through the origin of \mathbf{C}^n . As the first Chern class $c_1(X)$ vanishes, the total space X of the canonical bundle $\mathcal{O}_G(-n)$ is a non-compact Calabi–Yau manifold of dimension $d = k \cdot k' + 1$, where $k' = n - k$. The Grassmannian $G_{k,n}$ is the exceptional divisor of the blow up of a singularity \hat{X} reached in the limit of vanishing Kähler class. A more detailed description of this geometry is included in App. A. For the geometry under consideration, $\dim(\mathcal{H}_{cl}^{top}) = \sum_k h_c^{k,k}(X) = \chi_c(X) = \binom{n}{k} \equiv N$, where the subscript c refers to the cohomology with compact support.

3.1. The GLSM for the non-compact Calabi–Yau $X = \mathcal{O}_G(-n)$

The relevant GLSM is a $(2, 2)$ supersymmetric 2d gauge theory with gauge group $H = U(k)$, n chiral matter super fields X_α , $\alpha = 1, \dots, n$, in the fundamental representation and one extra super field P that transforms as $(\det M)^{-n}$, where $M \in U(k)$ acts on the fundamental representation [13]. The lowest components x_α^i , $i = 1, \dots, k$ and p of the matter super-fields parametrize the vacuum geometry of the 2d gauge theory. The D-term equations impose the following constraints on a supersymmetric vacuum:

$$D_j^i = \sum_\alpha x_\alpha^i \bar{x}_{j\alpha} - \delta_j^i (n |p|^2 + r) = 0. \quad (3.1)$$

Here r is the FI parameter for the $U(1)$ part of the gauge group, which is the imaginary part of the complexified Kähler class $t = \frac{\theta}{2\pi} + i r$.

For all non-zero values of the second term, the constraint (3.1) imposes that the x^i are k orthogonal vectors in \mathbf{C}^n of norm $n|p|^2 + r$. For $r > 0$ this norm is strictly non-zero and after dividing by the gauge transformations in $U(k)$, the gauge invariant information described by the k n -vectors x^i is a k -plane $\Lambda_k \subset \mathbf{C}^n$. This is the GLSM representation of the Grassmannian $G_{k,n}$ as the symplectic quotient $\mathbf{C}^{kn}/U(k)$ [13]. Moreover, as the extra field p transforms as a coordinate on the fiber of the bundle

$X = \mathcal{O}_G(-n)$, the total target space of the GLSM is the large volume phase of the non-compact Calabi–Yau $X = \mathcal{O}_G(-n)$ of dimension d .

For $r < 0$ the allowed vev’s include also the set $\{x_\alpha^i = 0 \ \forall i, \alpha\}$, while on the other hand p must be non-zero. Setting p equal to one by a $U(1) \subset U(k)$ transformation, leaves a remaining gauge invariance Γ' generated by the $U(k)$ matrices M with $\det(M)^n = 1$. The group Γ' contains a continuous subgroup $SU(k) \subset \Gamma'$ which can be divided out by passing to the Plücker embedding of $G_{k,n}$. This is described in more detail in App. A. The resulting space is the d -dimensional non-compact Calabi–Yau \hat{Y} which is the cone over a system of quadrics in \mathbf{P}^{N-1} , divided by the action of the discrete group $\Gamma : y_k \rightarrow \omega y_k$, with $\omega^n = 1$. Here the y_k denote homogeneous coordinates on \mathbf{P}^{N-1} . In the original coordinates x_α^i , this discrete group $\Gamma \simeq \mathbf{Z}_n \subset \Gamma'$ may be described by the sequence

$$\mathbf{Z}_{kn} \rightarrow \Gamma \rightarrow SU(k),$$

where \mathbf{Z}_{kn} is generated by the matrix $\Omega = \tilde{\omega} \cdot \mathbf{1}_{k \times k}$ with $\tilde{\omega}^{kn} = 1$; it fulfills $\Omega^n \in SU(k)$. The gauge transformation generated by $\Omega \in U(k)$ acting on the matrix (x_α^i) from the left may be alternatively represented as a discrete $SU(n)$ transformation acting by the matrix $\tilde{\omega} \cdot \mathbf{1}$ from the right.

We proceed with a study of the boundary topological sector of the above GLSM. In a first step we consider the ring structure generated by the super-fields X_α^i . The extra field P does not introduce new sectors and it will be easy to implement it at the end.

3.2. The basis $\{R_a\}$ from \mathcal{R}_Δ^+

The lowest components x_α^i of the even super fields (Ψ_ν^+) at the boundary are the projections of the bosonic components in X_α^i . They are in the representation $(\underline{k}, \underline{n})$ of $U(k) \times U(n)$, where $U(k)$ refers to the gauge group, while the $U(n)$ acts as a global symmetry and has a trivial connection. Let ν denote the vector that specifies a $SU(m)$ Young tableau with rows of length ν_i and total number of boxes $|\nu| \equiv \sum_i \nu_i$. To avoid confusion, note that the labels ν in (2.1) are completely *general* labels for the boundary fields which may be identified with Young tableaux only for the specific topological boundary fields (Ψ_ν^+) . We drop also the boundary sector indices (a, b) in the following, as the relevant sector will be obvious from the context.

Acting with the ring \mathcal{R}_Δ^+ generated by the fields x_α^i on a ground state $\nu(R_1) = (0, \dots, 0)$ with $U(1) \subset U(k)$ charge m_+ , generates ground states $\Phi_\nu^+ \in \mathcal{H}_{op}^{top}$ which

are in $H = U(k)$ representations labeled by the $SU(k)$ Young tableaux¹⁰ ν . We assert that a finite basis $\{R_a\}$ of generators for \mathcal{H}_{op}^{top} may be chosen as a sequence of $\chi_c(X) = N = \binom{n}{k}$ elements labeled by Young tableaux $\{\nu\}$ with at most $k' = n - k$ columns and at most k rows,

$$\underline{\mathcal{R}_{\Delta}^+} \rightarrow \{R_a\} = \{\Phi_{\nu}^+ : \nu_1 \leq n - k, \nu_i = 0 \text{ for } i > k\}. \quad (3.2)$$

First note that the elements of $\{R_a\}$ will generate the charge lattice of the twisted RR gauge fields precisely if the “intersection form” $\chi_{ab}^+ = \langle R_a, R_b \rangle$ defined by the inner product (2.3)

$$\langle A, B \rangle = \sum_k (-1)^k \dim \text{Ext}^k(A, B), \quad (3.3)$$

is non-degenerate. In particular we may express a state $V \in \mathcal{H}_{op}^{top}$ as the formal integral linear combination $V = \sum_a \langle V, R_a \rangle (\chi^+)^{-1 ab} R_b$, which describes the twisted RR charges of V .

We will verify the non-degeneracy of the intersection form for the basis $\{R_a\}$ below. The meaning of the particular choice of N symmetrizations (3.2) is that the set $\{R_a\}$ will be orthogonal to the elements of the set $\{S^a\}$ constructed from the fermionic generators in \mathcal{R}_{Δ}^- in the next section. In the latter case the truncation to a specific list of N symmetrizations will be entirely fixed by the fermionic statistics of the generators.

The intersection form χ^+ is determined by counting the maps $\Phi_{\nu} \rightarrow \Phi_{\nu'}$ with the degree k identified with the fermion number of the map [6]. Let us first count the maps associated to the single generator x_1^i of \mathcal{R}_{Δ}^+ . As x_1^i is bosonic, the composition of maps of degree > 1 must be totally symmetric. Thus the contribution $\tilde{\chi}_{ab}^+$ of x_1^i to χ_{ab}^+ is

$$\tilde{\chi}_{ab}^+ = (N_{\sigma_{m_{ab}}})_{\nu_a}^{\nu_b}, \quad (3.4)$$

where ν_a (ν_b) denotes the Young tableau that labels the symmetrization of R_a (R_b) and $m_{ab} = |\nu^{(b)}| - |\nu^{(a)}|$. Moreover σ_m denotes the m -th totally symmetric product and the (N_{μ}) are the fusion coefficients (2.1) of the boundary chiral ring determined by the tensor product decomposition

$$\mu \otimes \nu = \sum_{\rho} (N_{\mu})_{\nu}^{\rho} \rho \quad (3.5)$$

¹⁰ The $U(1)$ charge of the state Φ_{ν} is fixed by ν and m_+ and will not be explicitly written. In fact the charge m_+ of the ground state $(0, \dots, 0)$ can be freely chosen at this point and corresponds to a choice for the closed string background.

in $U(\infty) \supset U(k)$. To obtain the full matrix χ_{ab}^+ we notice that the totality of maps from the x_α^i is the composition of the maps (3.5) from the individual x_α^i , and thus χ_{ab}^+ is simply the n -th power of $\tilde{\chi}_{ab}^+$:

$$\langle R_a, R_b \rangle = \chi_{ab}^+ = (\tilde{\chi}^{+n})_{ab} = \sum_{\mu} (N_{\mu})_{\nu_a}^{\nu_b} \cdot \dim_{U(n)}(\mu). \quad (3.6)$$

The second expression follows from an alternative counting of the maps of all x_α^i at the same time. Namely in addition to the totally symmetric maps there are now also maps corresponding to Young tableaux with $i \leq k$ boxes anti-symmetrized. The bosonic statistics of the x_α^i implies that the symmetrization of the global $U(n)$ index α coincides with that of the $U(k)$ index i , and thus the multiplicity of a map of a $U(k)$ symmetrization defined by the Young tableau μ is the dimension of the “same” representation μ in $U(n)$. In fact, $\chi_{ab}^+ = (\tilde{\chi}_{ab}^+)^n$ can be easily seen to coincide with Kapranov’s result¹¹ [9] for the relative Euler number for sheaves on Grassmannians; the relation will be explained in the next section.

That the matrix χ_{ab}^+ is invertible will be shown below by constructing its inverse. Note that if we order $\{R_a\}$ with increasing $|\nu^{(a)}|$, as we do in the following, then χ_{ab}^+ will be upper triangular.

3.3. The basis $\{S^a\}$ from \mathcal{R}_{Δ}^-

The lowest components ψ_α^i of the odd super-fields (Ψ_ν^-) arise from the projections of the fermions in the super-fields X_α^i . Acting with ring \mathcal{R}_{Δ}^- generated by the fields ψ_α^i on a “trivial” ground state with $U(1)$ charge m_- generates another set of ground states $\Phi_\nu^- \in \mathcal{H}_{op}^{top}$. The discussion is similar to the previous case for \mathcal{R}_{Δ}^+ with two major modifications: *i*) the fermionic statistics leads to a natural truncation to a finite basis of N elements; *ii*) the lower index α is no longer a global symmetry index but participates in a gauge transformation.

The fermionic statistics implies that the symmetrization ν of the index $\alpha = 1, \dots, n$ is combined with a symmetrization ν^* of the $U(k)$ index $i = 1, \dots, k$, where ν^* denotes the Young tableau transpose to ν . We may thus label the symmetrization by the Young tableau ν for α only. To proceed we note that the GLSM contains mass terms for k^2 out of the $k \cdot n$ fermions ψ_α^i . In fact the $k \cdot k'$ massless fermions ψ_α^i are described by the last term of the sequence

$$0 \rightarrow \text{End}(V) \rightarrow \mathbf{C}^n \times V \rightarrow W \rightarrow 0, \quad (3.7)$$

¹¹ A more readable account is given in ref. [16].

where V is the k -dimensional vector space on which $U(k)$ acts linearly. A convenient local gauge choice is $\psi_\alpha^i = 0$ for $\alpha = 1, \dots, k$, which leaves $k \cdot k'$ fermions ψ_α^i , $\alpha = k+1, \dots, n$ as the local generators for the ring \mathcal{R}_Δ^- .

From the above it follows that the action of \mathcal{R}_Δ^- on the state $\nu = (0, \dots, 0)$ of $U(1)$ charge m_- generates the N ground states Φ_ν^- specified by the $U(n)$ representations

$$\underline{\mathcal{R}_\Delta^-} : \quad \rightarrow \quad \{S^a\} = \{\Phi_\nu^- : \nu_1 \leq k, \nu_i = 0 \text{ for } i > n - k\}. \quad (3.8)$$

Note that these states carry in addition the representations ν^* w.r.t. to the $H = U(k)$ gauge symmetry.

The evaluation of the intersection form $\chi_-^{ab} = \langle S^a, S^b \rangle$ is similar as before. Specifically, the multiplicities for the maps from a single fermion $\psi_{\alpha=1}^i$ and for the totality of maps, weighted by fermion number, respectively, are

$$\begin{aligned} \tilde{\chi}_-^{ab} &= (-)^{m_{ab}} (N_{\epsilon_{m_{ab}}})_{\nu_a}^{\nu_b}, \\ \chi_-^{ab} &= (\tilde{\chi}_-^n)^{ab} = \sum_{\mu} (-1)^{|\mu|} (N_{\mu})_{\nu_a}^{\nu_b} \cdot \dim_{U(n)}(\mu). \end{aligned} \quad (3.9)$$

Here ϵ_m denotes the m -th totally anti-symmetric representation and, as before, $m_{ab} = |\nu_a| - |\nu_b|$.

3.4. Orthogonality and a relation to the bulk chiral ring \mathcal{R}_{cl}

To show that the intersection forms (3.6) and (3.9) are non-degenerate, we establish now the relation

$$\sum_b \tilde{\chi}_-^{ab} \tilde{\chi}_{bc}^+ = \sum_{\nu_b} (-)^{m_{ab}} (N_{\epsilon_{m_{ab}}})_{\nu_a}^{\nu_b} (N_{\sigma_{m_{bc}}})_{\nu_b}^{\nu_c} = \delta_{ac} \quad (3.10)$$

It implies that for a judicious choice of the ground states S^1 and R_1 , the twisted RR charges of the elements in $\{R_a\}$ and $\{S^a\}$ generated by the action of the rings \mathcal{R}_Δ^\pm are related by the linear transformation $S^{a*} = \chi_-^{ab} R_b$. This implies in turn the orthogonality relation

$$\langle S^{a*}, R_b \rangle = \delta_b^a. \quad (3.11)$$

The significance of this relation for the construction of the fractional branes was pointed out in [17]. The choice of base points for which the above relation is true is $m_- = -n - m_+$, as will be derived in the geometric phase below.

The proof of (3.10) for $k = 1$ and generalizations to weighted projective spaces has been given in [6]. For general k the relation can be understood by first noting

that the (N_μ) coincide with the structure constants of the (classical) cohomology ring, $H^*(G_{k,n})$; in other words, the (N_μ) form a matrix representation of $H^*(G_{k,n})$. This follows from their definition (3.5) in terms of $U(k)$ tensor products, in conjunction with the result of [18] that equates the $U(k)$ fusion rules with the cup product of the (quantum¹²) cohomology ring of the Grassmannians. It is known that the cohomology ring is generated by the Chern classes c_i , $i = 1, \dots, k$, and these are represented by the matrices (N_{σ_i}) associated with the fully symmetric representations. Moreover the normal Chern classes, $\bar{c}_{i'}$, are associated with the totally anti-symmetric Young tableaux¹³ and are represented by $(-)^{i'} (N_{\epsilon_{i'}})$, $i' = 1, \dots, k'$. The orthogonality relation (3.10) is therefore nothing but:

$$\left(\sum_{i'} (-1)^{i'} N_{\epsilon_{i'}} \right) \cdot \left(\sum_i N_{\sigma_i} \right) = \left(1 + \bar{c}_1 + \dots + \bar{c}_{k'} \right) \cdot \left(1 + c_1 + \dots + c_k \right) = \mathbf{1}_{k \times k},$$

It is thus simply a matrix representation of the equation that states the triviality of the bundle $E \oplus F = \mathbf{C}^n$.

3.5. Relation to $N = 2$ coset models

Note that $\tilde{\chi}$ has showed up in previous work [20], in the context of the $N = 2$ super-conformal coset models based on $G_{k,n}$. In that work the open string index [21] $\chi_{CFT}^{ab} \equiv \text{Tr}_{a,b}(-1)^F$ of the coset boundary states was computed and found to be given in terms of $U(k)$ fusion coefficients. Choosing a minimal, non-extended basis of the boundary states for which the fusion coefficients are upper-triangular matrices, the index coincides with $\tilde{\chi}_-^{ab}$ as given in (3.9). This means that the CFT intersection index for the $N = 2$ superconformal coset models has a very close relationship to the intersection form for sheaves on $G_{k,n}$, i.e.,

$$\chi_- = (\chi_{CFT})^n. \quad (3.12)$$

This expresses a structural isomorphism between the $N = 2$ sigma model on $G_{k,n}$ and the $N = 2$ coset model based on $G_{k,n}$, which has been known since a long time as far as topological bulk physics is concerned [11]. That is, the (appropriately perturbed) chiral rings of these models are isomorphic, even though the charges of the chiral fields are different. The identity (3.12) may be viewed as the reflection of this in the boundary sector of these models. It says that the boundary rings are isomorphic up to multiplicities.

¹² In the present situation we encounter the classical cohomology ring, which amounts to truncating the $U(k)$ fusion coefficients to upper triangular matrices.

¹³ For details see e.g. [19], chapter III.

3.6. The extra field P

So far we have neglected the generators in \mathcal{R}_Δ associated to the extra field P that adds the non-compact direction of the fiber of $\mathcal{K} = \mathcal{O}_G(-n)$. As the Hilbert space \mathcal{H}_{op}^{top} is related to the compact part of the non-compact Calabi-Yau X , the field P does not add new ground states. However P generates new maps and thus changes the intersection forms χ_{ab}^+ and χ_{ab}^- . As the field P is associated to the canonical bundle, the additional maps follow most easily from Serre duality

$$[H^k(X, V)]^* \simeq H^{d-k}(X, V^* \otimes \mathcal{K}), \quad (3.13)$$

with the result that

$$\chi^+(X) = \chi^+ + (-)^n \chi^{+T}, \quad \chi_-(X) = \chi_- + (-)^n \chi_-^T. \quad (3.14)$$

In fact these expressions agree with those derived in [6] for the intersection form of the restrictions of the same sheaves to a compact hypersurface Y embedded in the exceptional divisor of X .

4. The large volume phase: exceptional sheaves on $\mathcal{O}_G(-n)$

A variation of the FI parameter r in (3.1) interpolates between different phases of the 2d gauge theory, and in particular connects the geometric quotient at small volume continuously to the large volume phase that describes the resolution of it. In the large volume phase the matter fields X_α^i parameterize the exceptional divisor $E = G_{k,n}$ of the resolution and are acted upon by the full $U(k)$ group. Moreover the natural interpretation of the topological ground states in the boundary sector is in terms of K-theory [3]. As stated in sect. 2, the relevant objects are the large volume duals, $\{R_a^\infty\}$ and $\{S_\infty^a\}$, of the two bases $\{R_a\}$ and $\{S^a\}$ constructed from the chiral boundary ring at small volume. We will now verify the properties of $\{R_a^\infty\}$ and $\{S_\infty^a\}$ as predicted by the conjecture 1.

4.1. Identification of the dual bases $\{R_a^\infty\}$ and $\{S_\infty^a\}$

There are two universal bundles over $G_{k,n}$, namely the universal sub-bundle

$$E = \{(\Lambda_k, z) \in G_{k,n} \times \mathbf{C}^n : z \in \Lambda_k\}, \quad (4.1)$$

which has the k plane $\Lambda_k \in G_{k,n}$ as its fiber, and the universal quotient bundle F defined by the exact sequence

$$0 \longrightarrow E \longrightarrow \mathbf{C}^n \longrightarrow F \longrightarrow 0. \quad (4.2)$$

Very importantly, the gauge bundle $U(k)$ of the GLSM is identified in the IR with the dual E^* of the universal sub-bundle [18]. Therefore the fields x_α^i related to \mathcal{R}_Δ^+ are sections of E^* on the resolution X and the dual basis $\{R_a^\infty\}$ is the collection of bundles

$$\{R_a^\infty\} = \{\Sigma^\nu E^*\}, \quad (4.3)$$

where ν runs over the N Young tableaux in eq.(3.2), and $\Sigma^\nu V$ denotes the symmetrization of the product bundle $V^{\otimes|\nu|}$ defined by the Young tableau ν .

As for the second collection $\{S_\infty^a\}$, we may replace (3.7) by

$$0 \longrightarrow E \otimes E^* \longrightarrow \mathbf{C}^n \otimes E^* \longrightarrow F \otimes E^* \longrightarrow 0,$$

where the last term is in fact the tangent bundle on $G_{k,n}$, $\Omega^* = F \otimes E^*$. To determine the large volume continuation of the ground states obtained from products of the fermions ψ_α^i in \mathcal{R}_Δ^- , we have to take into account also their non-trivial representation under the gauge group $U(k)$. In total, the states take values in the space of sections of wedge products of the tangent bundle twisted by E^* , $\{S_\infty^a\} \subset \wedge^i(E^* \otimes \Omega^*) = \wedge^i(E^* \otimes E \otimes F)$. A simplification occurs as the relevant space is, by construction, the subset generated by the group of global, holomorphic sections. However $h^0(E \otimes E^*) = 1$ with the single global section corresponding to the singlet in the tensor decomposition of $\underline{k} \otimes \underline{k}$. Therefore the space of ground states generated by the global sections is

$$\{S_\infty^a\} = \{\Sigma^\nu F\}, \quad (4.4)$$

where ν runs over the N Young tableaux in eq.(3.8).

As an independent check of these identifications¹⁴, one may use the r.h.s. of (2.3) to verify that the intersection forms χ^+ (3.6) and χ_- (3.9), as determined from the structure constants of the boundary ring \mathcal{R}_Δ , satisfy¹⁵:

$$\chi_{ab}^+ = \int_X \text{ch}(R_a^{\infty*}) \text{ch}(R_b^\infty) \text{td}(X), \quad \chi_-^{ab} = \int_X \text{ch}(S_\infty^{a*}) \text{ch}(S_\infty^b) \text{td}(X). \quad (4.5)$$

¹⁴ For Grassmannians $G_{k,n}$ and more general flag manifolds, the group theoretical and geometric descriptions are also related by the Bott-Borel-Weil theorem.

¹⁵ In general the expressions (3.6) and (3.9) as determined from the structure constants of the boundary ring, agree with the geometric integrals only after taking into account the appearance of additional massless fields in the large volume phase [6].

With eqs.(4.3), (4.4), we have precisely recovered Kapranov's exceptional collections of sheaves [9] on $G_{k,n}$, which represent free generators for the derived category $\mathcal{D}^b(G)$. In physics terms this means that any D-brane on $G_{k,n}$ can be written in terms of a bounded complex involving only either the sheaves R_a^∞ , or the S_∞^a . From the above one may also easily see that the orthogonality relation (3.11) will hold for $R_N = S^{N*}$ which implies the previously mentioned condition $m_- = -n - m_+$.

4.2. Helices of exceptional sheaves on $G_{k,n}$

It has been observed in [6,22,23] that the large volume versions $\{R_a^\infty\}$ and $\{S_\infty^a\}$ of the ‘‘McKay bases’’ for weighted projective spaces represent foundations of a helix structure on the exceptional divisor E of the resolution $X \rightarrow \hat{X}$. We briefly describe now how these results extend to the present case (which is granted given the work of [9]) and outline the property *ii*) of the conjecture. For details on the definitions of a helix and references we refer to [16]. In brief, a helix \mathcal{H} of period N is an infinite series of exceptional sheaves such that N consecutive elements represent an exceptional collection. An exceptional sheaf E is defined by $\text{Ext}^0(E, E) = \mathbf{C}$, $\text{Ext}^k(E, E) = 0$, $k > 0$ and an exceptional collection \mathcal{E} is an ordered collection of exceptional sheaves with $\text{Ext}^k(E_a, E_b) = 0$ for all $a \neq b$ and k , except possibly for a single value of k if $a < b$.

That the collections $\{R_a^\infty\}$ and $\{S_\infty^a\}$ in (4.3) and (4.4) are exceptional on $G_{k,n}$, serve as a foundation for a helix structure and provide, respectively, free generators for $\mathcal{D}^b(G)$ has been shown in [9]¹⁶. In particular the exceptionality is reflected in the upper triangular form of the matrices χ^+ and χ^- and their unit diagonal.

Whereas the definition of the bundles R_a^∞ as tensor products of the vacuum bundle E is canonical, the same is not in general true for the dual bundles S^a . Of course, in the present case, the bundles S^a have been more directly identified as certain powers of the quotient bundle F . However to construct the pull-backs of the sheaves R_a and S^a to the non-compact space X , it is essential to look at a canonical relation between them given in terms of a certain operation on exceptional collections, the so-called mutations.

A mutation is an operation on two neighbors of an exceptional collection that produces a new exceptional collection. There are two possible cases acting as

¹⁶ To be precise, our definition of the objects S^a differs by a factor $(-1)^k$ from that in [9]. This is related to the fact that a brane obtained from an odd number of fermions is interpreted as an anti-brane, see e.g [2].

$(E_a, E_{a+1}) \rightarrow (E_{a+1}, \underline{R}E_a)$ and $(E_{a-1}, E_a) \rightarrow (\underline{L}E_a, E_a)$, called a right and left mutation, respectively. We refer to [24] for the details on the definitions. The two bases $\{R_a^\infty\}$ and $\{S_a^\infty\}$ are related by the special series of right mutations \underline{P} [6,22,23]:

$$\underline{P}: \{R_1^\infty, \dots, R_N^\infty\} \longrightarrow \{R_N^\infty, \underline{R}R_{N-1}, \dots, \underline{R}^{N-1}R_1^\infty\} = \{S_\infty^{N+1-a*}\}. \quad (4.6)$$

In fact the relations $S_\infty^{a*} = \underline{R}^{N-a} R_a^\infty$ derive from the tautological sequence (4.2). For the foundation with $R_N^\infty = \mathcal{O}$, we have $R_{N-1}^\infty = \det(E^*)^{-1} \otimes (\wedge^{k-1} E^*) = E$ and the right mutation defined as

$$0 \longrightarrow R_{N-1}^\infty \longrightarrow \mathrm{Hom}(R_{N-1}^\infty, R_N^\infty)^* \otimes R_N^\infty = \mathbf{C}^n \longrightarrow \underline{R}R_{N-1}^\infty \longrightarrow 0$$

coincides with (4.2). This recovers our previous result $S_\infty^{N-1*} = \det(F)^{-1} \otimes (\wedge^{n-k-1} F^*) = F$.

4.3. McKay bases on the non-compact space X from the helix on E

We complete now the construction of a McKay correspondence by extending the definition of the sheaves R_a and S^a to the total space X . As we will only use general properties of the helix on E , the results of this section are more general and apply to any complete resolution $E \rightarrow X \rightarrow \hat{X}$. The corresponding sheaf collections on X will be denoted by $\{\mathcal{R}_a\}$ and $\{\mathcal{S}^a\}$ and provide, similarly as in [8], generators for the K-theory groups $K(X)$ and $K_c(X)$, respectively.

The sheaves \mathcal{R}_a will be simply defined as the pull-backs of R_a by the restriction map $\pi: X \rightarrow E$, where e.g. $E = G_{k,n}$ as in the previous sections. Inspired by an observation of Tomasiello for $E = \mathbf{P}^n$ [22], we will define the \mathcal{S}^a through a certain complex on X that reduces to the mutation \underline{P} on the compact exceptional divisor E . In fact the main difference between the collections on E and on X is that there are no exceptional collections on X because of $c_1(X) = 0$. Indeed a defining property of a helix of coherent sheaves is $r^{N-1}E_k = E_{k+N} = E_k(\mathcal{K}^*)$, where \mathcal{K}^* is the anti-canonical bundle. Thus the definition of the helix collapses for $\mathcal{K}^* = \mathcal{O}$ and one expects instead some kind of *cyclic* structure with period N . Indeed Serre duality implies that the sheaves \mathcal{R}_a and \mathcal{S}^a on X have the non-zero extension groups:

$$\mathrm{Ext}^k(\pi^* E_a, \pi^* E_b) = \begin{cases} \mathbf{C}, & a = b, k = 0, d, \\ \mathrm{Ext}^0(E_a, E_b), & a < b, k = 0, \\ \mathrm{Ext}^d(E_a, E_b), & a > b, k = d, \\ 0, & \text{else.} \end{cases} \quad (4.7)$$

Starting from $\mathcal{R}_a = \pi^* R_a$, and with $\mathcal{R}_a(K^*) = \mathcal{R}_a \otimes \mathcal{K}^*(E)$, we define the sheaves \mathcal{S}^{a*} by the first of the following two, closely related sequences:

$$\begin{aligned} \mathcal{S}^{a*} : \mathcal{R}_a(K^*) &\rightarrow a_{ba}^1 \mathcal{R}_b(K^*) \rightarrow a_{ba}^2 \mathcal{R}_b(K^*) \rightarrow \dots \rightarrow a_{ba}^{N-1} \mathcal{R}_b(K^*) \rightarrow \mathcal{R}_a(K^*) \\ 0 \rightarrow \mathcal{R}_a &\rightarrow \tilde{a}_{ba}^1 \mathcal{R}_b \rightarrow \tilde{a}_{ba}^2 \mathcal{R}_b \rightarrow \dots \rightarrow \tilde{a}_{ba}^{N-1} \mathcal{R}_b \rightarrow \mathcal{R}_a(K^*) \rightarrow 0. \end{aligned} \quad (4.8)$$

A crucial point is that in the first sequence, the sum is over the finite set $b = 1, \dots, N$, reflecting the cyclic structure of X , while in the second sequence the sum is over all $b \in \mathbf{Z}$, reflecting the infinite helix structure on E . Accordingly, the coefficients a_{ba}^m and \tilde{a}_{ba}^m are defined as

$$\begin{aligned} \tilde{a}_{ba}^m &= \begin{cases} \dim \operatorname{Hom}(\underline{R}^{m-1} R_a, R_b) & b = m + a + 1, \\ 0 & \text{else} \end{cases}, \\ a_{ba}^m &\equiv \begin{cases} \tilde{a}_{ba}^m & b > a \\ \tilde{a}_{b+Na}^m & b < a \end{cases}. \end{aligned} \quad (4.9)$$

In particular the \tilde{a}_{ba}^m describe the morphisms on E and with this definition, the second sequence is a pull-back to X of the exact sequence for the identity $r^{N-1} R_a = R_a \otimes \mathcal{K}^*(E)$. On the other hand, the a_{ba}^m describe the cyclic structure on the total space X induced by the extra Ext 's in eq.(4.7).

Reducing to K -theory classes, one may then use the second sequence in (4.8) to rewrite

$$\begin{aligned} \mathcal{S}^{a*} &= \sum_{m=0}^N \sum_{b=1}^N (-)^{m-1} a_{ba}^m \mathcal{R}_b(\mathcal{K}^*) \\ &= \sum_{m=0}^N \sum_{b=1}^N (-)^m \tilde{a}_{ba}^m (\mathcal{R}_b - \mathcal{R}_b(\mathcal{K}^*)) \\ &= \sum_{m=0}^N \sum_{b=1}^N (-)^m \tilde{a}_{ba}^m R_b = \underline{R}^{N-a} R_a \\ &= \mathcal{S}^{a*}, \end{aligned} \quad (4.10)$$

where we have defined $\tilde{a}_{ba}^m = \delta_{ba}$ for $m = 0, N$ and we have used $\mathcal{R}_b - \mathcal{R}_b(\mathcal{K}^*) = \mathcal{R}_b|_E = R_b$. The above relation $\mathcal{S}^a = \mathcal{S}^a|_E = S^a$ shows that the K -theory classes generated by the collection $\{\mathcal{S}^a\}$ are in the compact K -theory group $K_c(X)$.

Eqs.(4.8) and (4.9) provide a general definition of the collections $\{\mathcal{R}_a\}$ and $\{\mathcal{S}^a\}$ on the non-compact space X which is the generalization of the ‘‘McKay bases’’ of Ito and Nakajima [8]. It is based solely on the helix structure on the exceptional divisor E and the above argument implies that the collections $\{\mathcal{R}_a\}$ and $\{\mathcal{S}^a\}$ generate the K -theory groups $K(X)$ and $K_c(X)$, respectively.

4.4. Monodromies and the D0-brane

We finish the section with two further comments. The first concerns the monodromy group \mathcal{G} of the naive, complexified Kähler moduli space. It is a subgroup of the invariance group of the intersection form $\chi_-(X)$. E.g. for the ADE quotient singularities \mathbf{C}^2/Γ , the invariance groups \mathcal{G} are the corresponding Weyl groups. For the quotients $O_{G_{1,n}}(-n)$ the local monodromy group in fact coincides with Γ , namely $\mathcal{G} = \mathbf{Z}_n$; a similar statement holds for the generalization to weighted projective spaces. As a consequence, the foundations $\{R_a^\infty\}$ and $\{S_\infty^a\}$ for weighted projective spaces \mathbf{WP}^n may be generated from a single monodromy $T_\infty = AT$ on a hypersurface Y embedded in \mathbf{WP}^n [6,23], where A is the monodromy around the Gepner point and T the conifold monodromy; moreover a certain power T_∞ generates the shift of the complexified Kähler class by one. The same is not true for the singularities $O_{G_{k,n}}(-n)$ for $k > 1$, as a consequence of the fact that the monodromy at small volume is not Abelian. In fact the invariance group of the intersection form $\chi_-(X)$ is a subgroup of the Weyl group of $U(n)$. It would be interesting to obtain a collection of generating mutations also in this case.

Secondly, let us identify the most fundamental bound-state of the fractional D-branes S^a , namely the D0-brane, or the class of a point on X . Its K-theory class is given by the linear combination¹⁷

$$[D0] = \sum_a r_a S^a, \quad r_a = \text{rank}(R_a). \quad (4.11)$$

From the quotient construction of gauge theories described in [25] we expect therefore that the D0 brane may be obtained as a one-dimensional branch of the moduli space of a $\prod_a U(r_a)$ gauge theory, with bi-fundamentals in the representations specified by the intersection form $\chi_-(X)$.

5. A case study: Quiver and fractional D-branes for the canonical bundle $O_{G_{2,5}}(-5)$

As an illustration of the above concepts and the effectiveness and simplicity of the approach, we present the determination of the McKay bases, or the spectrum of the fractional branes, for the non-compact Calabi–Yau $O_{G_{2,5}}(-5)$.

The bases $\{R_a\}$ and $\{S^a\}$ obtained from the boundary ring \mathcal{R}_Δ carry the representations specified in (3.2) and (3.8), respectively. The structure constants (3.4)

¹⁷ An alternative, in some sense minimal representation, is discussed in App. B.

of the boundary ring then immediately determine the intersection forms $\tilde{\chi}_-$ and $\chi_- = (\tilde{\chi}_-)^5$:

$\tilde{\chi}_-$	\cdot	\square	$\begin{smallmatrix} \square \\ \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \end{smallmatrix}$
\cdot	1	-1	0	1	0	0	0	0	0	0
\square	0	1	-1	-1	0	1	0	0	0	0
$\begin{smallmatrix} \square \\ \square \end{smallmatrix}$	0	0	1	0	-1	-1	1	0	0	0
$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$	0	0	0	1	0	-1	0	1	0	0
$\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}$	0	0	0	0	1	0	-1	0	0	0
$\begin{smallmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{smallmatrix}$	0	0	0	0	0	1	-1	-1	1	0
$\begin{smallmatrix} \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \end{smallmatrix}$	0	0	0	0	0	0	1	0	-1	0
$\begin{smallmatrix} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{smallmatrix}$	0	0	0	0	0	0	0	1	-1	1
$\begin{smallmatrix} \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square \end{smallmatrix}$	0	0	0	0	0	0	0	0	1	-1
$\begin{smallmatrix} \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \end{smallmatrix}$	0	0	0	0	0	0	0	0	0	1

χ_-	\cdot	\square	$\begin{smallmatrix} \square \\ \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \end{smallmatrix}$
\cdot	1	-5	10	15	-10	-40	45	50	-75	50
\square	0	1	-5	-5	10	25	-50	-40	95	-75
$\begin{smallmatrix} \square \\ \square \end{smallmatrix}$	0	0	1	0	-5	-5	25	10	-50	45
$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$	0	0	0	1	0	-5	10	15	-40	50
$\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}$	0	0	0	0	1	0	-5	0	10	-10
$\begin{smallmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{smallmatrix}$	0	0	0	0	0	1	-5	-5	25	-40
$\begin{smallmatrix} \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \end{smallmatrix}$	0	0	0	0	0	0	1	0	-5	10
$\begin{smallmatrix} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{smallmatrix}$	0	0	0	0	0	0	0	1	-5	15
$\begin{smallmatrix} \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square \end{smallmatrix}$	0	0	0	0	0	0	0	0	1	-5
$\begin{smallmatrix} \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \end{smallmatrix}$	0	0	0	0	0	0	0	0	0	1

Inclusion of the P field leads to the anti-symmetrized matrices $\chi^+(X)$ and $\chi_-(X)$ in eq. (3.14), respectively.

Upon interpolation, these small-volume boundary ring data map to the large volume data which have a geometrical interpretation. Specifically, the elements of the large volume bases $\{R_a^\infty\}$ and $\{S_\infty^a\}$ are the sheaves $\Sigma^\nu E^*$ and $\Sigma^\nu F$, resp., where ν runs over the same Young tableaux as above. The Chern character of these bundles may be expressed in terms of $c(E) = 1 - c_1 + c_2$ by the standard formulae [26]. Using the relations $c_2^2 - 3c_2c_1^2 + c_1^4 = 0$, $-3c_1c_2^2 + 4c_2c_1^3 - c_1^5 = 0$, the Todd class

$$\begin{aligned}
\text{td}(G) = & 1 + \frac{5}{2}c_1 + \frac{1}{12}(36c_1^2 + c_2) + \frac{5}{24}c_1(11c_1^2 + c_2) \\
& + \frac{1}{720}(897c_1^4 + 179c_2c_1^2 - 3c_2^2) + \frac{1}{96}c_1(49c_1^4 + 18c_2c_1^2 - c_2^2) \\
& + \frac{1}{60480}(9848c_1^6 + 6029c_2c_1^4 - 746c_2^2c_1^2 - 72c_2^3)
\end{aligned}$$

and $\int_X c_2^3 = 1$ one may verify that the integrals (4.5) agree with (5.1). This confirms the advertised correspondence between the small volume (group theoretical) and large volume (K theoretical) data.

From the intersection data we may draw the quiver graph¹⁸ in Fig.1 associated to the exceptional collection $\{S_\infty^a\}$, with a node for each ground state S^a and a link

¹⁸ A very similar diagram appeared in ref. [20] for boundary states of a Kazama-Suzuki coset model, the difference only being in the multiplicities of the links. As explained before, this reflects the structural isomorphism between the coset model and the sigma model on the Grassmannian.

between nodes representing a fermionic zero mode contributing to the index χ_- . The links indicate the basic maps generated by the fundamental anti-symmetric representations contributing to $\tilde{\chi}_-$. Specifically fat links denote the five maps generated by the sections ψ_α^i , and thin ones the fifteen maps generated by $\psi_{(\alpha}^i \psi_{\beta)}^j$. Composing these basic maps according to (3.9) leads to further links in the diagram, like for example the dashed ones. Taking the P field into account adds the links with reversed arrows.¹⁹

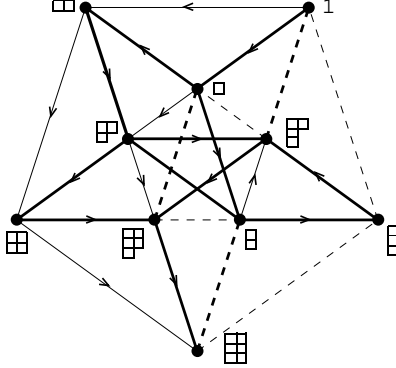


Fig.1: Quiver graph associated with the sheaves $\Sigma^\alpha F$ on $G_{2,5}$

According to [5], intersection diagrams such as the one in Fig.1 can also be viewed in terms of solitons of a mirror Landau-Ginzburg theory. In fact, a general relation between collections of exceptional sheaves on a Fano variety and special Lagrangian cycles in a LG theory has been derived in [5] using local mirror symmetry. This gives another powerful description of the system of D-branes in terms of the complex deformations of a holomorphic superpotential W . We include a discussion of these aspects for the interested reader in App. B.

6. Related quotient singularities and compact Calabi–Yau 3-folds

The canonical bundle $\mathcal{O}_G(-n)$ is just one of a larger class of non-compact Calabi–Yau singularities X that share the same compact homology $H_*(X) = H_*(G_{k,n})$. Specifically the condition $c_1(X) = 0$ is equivalent to the vanishing beta function for the FI parameter in the GLSM [13], and thus any choice of additional matter fields

¹⁹ Note that although we have drawn the nodes in a \mathbf{Z}_5 symmetric manner, the links are not \mathbf{Z}_5 symmetric; this is in contrast to the quiver for $\mathcal{O}_{\mathbf{P}^4}(-5)$. The location of the nodes gets a meaning in the W -plane of the mirror LG model discussed in App. B, where we also describe a modified quiver with \mathbf{Z}_5 symmetric links.

that implies the vanishing of the beta function leads to a non-compact Calabi–Yau X with exceptional divisor $G_{k,n}$ ²⁰.

Let us consider only a slight generalization, where the field P is replaced by m fields P_j of $U(1) \subset U(n)$ charges $-q_j$ with $\sum_j q_j = n$. There is a phase of the gauge theory where the total space is that of the bundle $\oplus_j \mathcal{O}_G(-q_j)$, which is a $k \cdot (n-k) + m$ dimensional non-compact Calabi–Yau. To describe the quotient singularity, consider the Plücker embedding $\mathbf{P}^N|_{\{Q_i=0\}}$ of $G_{k,n}$, where $\{Q_i\}$ is a system of quadrics. The singularity may be described as follows. Consider space $\mathbf{C}^N|_{\{Q_i=0\}} \times_{\mathbf{C}^*} \mathbf{C}^m$, where the \mathbf{C}^* acts as $(y_i; p_j) \rightarrow (\omega y_i; \omega^{-q_j} p_j)$ on the coordinates of the two factors. Dividing by \mathbf{C}^* , a solution of the D-terms for $r < 0$ implies that the projection to the second factor is a $\mathbf{WP}_{\{q_j\}}^{m-1}$. The fiber F_p of this projection at a point $p \in \mathbf{WP}_{\{q_j\}}^{m-1}$ is \mathbf{C}^N/Γ , where $\Gamma \subset U(1)$ is the subgroup of $U(1)$ that fixes p . From the discussion in sect. 3.6, the collections of ground states $\{R_a\}$ and $\{S^a\}$ is independent of the choice “ P -fields”, however the intersection forms $\chi^\pm(X)$ depend on it.

Although the K-theory of these non-compact spaces might be interesting to study, let us discuss *compact* complete intersection Calabi–Yau’s $Y \subset G_{k,n}$ defined by the intersection of the zero locus of the m sections s_j of $\mathcal{O}_G(q_j)$. In the GLSM the constraints are described by the addition of a superpotential $\sum_{j=1}^m P_j S_j(X_\alpha^i)$, where the S_j are the super-fields with $s_j(x_\alpha^i)$ as lowest components. The complex dimension of the CICY Y is then $k \cdot (n-k) - m$.

In particular $m = 1$, $q_j = n$ describes the single hypersurface. Note that in this case the extension groups in eq.(4.7) describe also the restriction to the hypersurface $Y \subset G \subset X$. The sheaves $R_a^\infty|_Y$ and $V^a = S_\infty^a|_Y$ thus give rise to automorphisms, or Fourier-Mukai transforms, on $K(Y)$:

$$V \rightarrow p_{2*}(p_1^*V \otimes \Delta_{E_a}), \quad V \in K(Y), \quad E_a \in \{R_a^\infty|_Y, S_\infty^a|_Y, \}$$

where Δ_{E_a} is the kernel of the map $E_a \boxtimes E_a^* \rightarrow \mathcal{O}_\Delta$ defined on the direct product $Y \times Y$ and p_i are the projections on its i -th factor.

From the point of string theory the most interesting case is $\dim_{\mathbf{C}}(Y) = 3$, as Y may then serve as a compactification manifold for the ten-dimensional string to four

²⁰ In general, however, the resolution X will be still singular and may or may not allow further resolutions to a smooth space.

dimensions. It turns out that there are only six choices²¹ of integers (k, n) that lead to an ambient space $G_{k,n}$ different from \mathbf{P}^{n-1} :

	$\chi(Y)$	$\int_Y K^3$	$\int_Y c_2 K$
$G_{2,4}[4]$	-176	8	56
$G_{2,5}[1, 1, 3]$	-150	15	66
$G_{2,5}[1, 2, 2]$	-120	20	68
$G_{2,6}[1, 1, 1, 1, 2]$	-116	28	76
$G_{2,7}[1, 1, 1, 1, 1, 1, 1]$	-98	42	84
$G_{3,6}[1, 1, 1, 1, 1, 1]$	-96	42	84

(6.1)

As in [17], D-branes on Y may then be obtained by restriction of the D-branes on X to Y . However the restriction map is *not* good in general as a basis for $\mathcal{D}^b(X)$ does not necessarily generate $\mathcal{D}^b(Y)$. A simple example is the complete intersection model $\mathbf{P}^5[2, 4]$ for $Y = G_{2,4}[4]$ obtained from the Plücker embedding. It is easy to see that the restriction of the exceptional collection $\mathcal{O}_P(k)$, $k = -5, \dots, 0$ for \mathbf{P}^5 to Y generates only a sub-lattice of $H^*(Y)$ by its Chern classes²². Modulo these questions, the calculation of the D-brane spectrum obtained by restricting to the complete intersections is straightforward.

As an example²³ let us consider the complete intersection $Y = G_{2,5}[1, 2, 2]$. The intersection form $\chi_-(Y)^{ab} = \langle V^a, V^b \rangle_Y$ for the restrictions $V^a = S^a|_Y$ is

$$\chi_-(X) = \begin{pmatrix} 0 & -5 & 10 & 16 & -9 & -40 & 40 & 52 & -65 & 38 \\ 5 & 0 & -3 & -10 & 10 & 16 & -49 & -30 & 92 & -65 \\ -10 & 3 & 0 & 20 & -5 & -8 & 26 & 0 & -49 & 40 \\ -16 & 10 & -20 & 0 & 2 & 80 & 0 & -104 & -30 & 52 \\ 9 & -10 & 5 & -2 & 0 & 0 & -5 & 2 & 10 & -9 \\ 40 & -16 & 8 & -80 & 0 & 0 & -8 & 80 & 16 & -40 \\ -40 & 49 & -26 & 0 & 5 & 8 & 0 & -20 & -3 & 10 \\ -52 & 30 & 0 & 104 & -2 & -80 & 20 & 0 & -10 & 16 \\ 65 & -92 & 49 & 30 & -10 & -16 & 3 & 10 & 0 & -5 \\ -38 & 65 & -40 & -52 & 9 & 40 & -10 & -16 & 5 & 0 \end{pmatrix}$$

The rank of this matrix is $\chi(Y) = 4$, equal to the number of periods of Y , and in fact it is easy to verify that the classes of the V^a generate the K-theory group $K(Y)$ over

²¹ We use the notation $G_{k,n}[q_1, q_2, \dots, q_m]$ for a complete intersection of m hypersurfaces of degree q_j . The mirror maps for the threefolds in the table have been studied in ref. [27].

²² A related fact is that the GLSM with target space $\mathbf{P}^5[2]$ is not equivalent to that with target space $G_{2,4}$.

²³ The results for the other cases are available upon request.

the integers. We may then proceed further and express the fractional branes V^a in the integral basis of symplectic charges \vec{Q} by a comparison of the central charges [4][28]:

$$Z(A) = - \int_Y e^{-J} \text{ch}(A) \sqrt{\text{td}(Y)} = \vec{Q} \cdot \vec{\Pi}, \quad (6.2)$$

where $\vec{\Pi} = (2\mathcal{F} - t\partial_t F, \partial_t \mathcal{F}, 1, t)^T$ is the period vector of Y with \mathcal{F} the prepotential for the special geometry of the complexified Kähler moduli space of \mathcal{M}_Y . The polynomial piece of \mathcal{F} which fixes the charges \vec{Q} may be determined from the topological data of the Calabi–Yau Y as in sect. 9.3. of [6]:

$$\mathcal{F} = -\frac{10}{3}t^3 + \frac{17}{6}t + \text{const.} .$$

From this we obtain the following symplectic charges $\vec{Q}^a = (Q_6, Q_4, Q_0, Q_2)^T(V^a)$ of the V^a :

$$\vec{Q}^a = \begin{pmatrix} -1 & 3 & -3 & -6 & 1 & 8 & -3 & -6 & 3 & -1 \\ 2 & -5 & 4 & 8 & -1 & -8 & 2 & 4 & -1 & 0 \\ -38 & 65 & -40 & -52 & 9 & 40 & -10 & -16 & 5 & 0 \\ -40 & 78 & -48 & -80 & 10 & 48 & -8 & 0 & -2 & 0 \end{pmatrix}$$

Acknowledgments: We thank M. Reid for email correspondence.

Appendix A. Geometry of the non-compact Calabi–Yau $X = \mathcal{O}_G(-n)$

We consider the total space of the canonical bundle $X = \mathcal{O}_G(-n)$, where $G = G_{k,n}$ is the Grassmannian parametrizing k -planes Λ_k through the origin of \mathbf{C}^n . The manifold X represents the blow up $X \rightarrow \hat{X}$ of a $d = k(n - k) + 1$ -dimensional Calabi–Yau singularity \hat{X} reached in the limit of vanishing Kähler volume of $G_{k,n}$. E.g. for $k = 1$, $\hat{X} = \mathbf{C}^n/\mathbf{Z}_n$ and $G_{1,n} = \mathbf{P}^{n-1}$ is the exceptional divisor of the blow up of \hat{X} .

For $k > 1$ the Calabi–Yau X and its singular limit \hat{X} may be described as follows. The k -plane $\Lambda_k \in \mathbf{C}^n$ may be represented by k linearly independent vectors $x^i \in \mathbf{C}^n$, $i = 1, \dots, k$. The $n \cdot k$ components x_α^i , $\alpha = 1, \dots, n$ provide homogeneous coordinates on $G_{k,n}$ which define local coordinates after dividing by the group $GL(k)$ that fixes Λ_k . The manifold $G_{k,n}$ may be embedded into \mathbf{P}^{N-1} with $N = \binom{n}{k}$ via the global sections of the ample line bundle $\mathcal{O}_G(1)$. More explicitly, the N homogeneous coordinates $y_{\{r\}}$ of \mathbf{P}^N are given by the determinants of the N $k \times k$ minors of the $k \times n$ matrix x_α^i which represent N global sections of $\mathcal{O}(1)$. The image $\varphi(G) \subset \mathbf{P}^{N-1}$ under this embedding is given by $\binom{n}{k+1}$ quadratic relations Q_i of rank $N - d$. The embedding φ is well-known as the Plücker embedding [29].

For large positive Kähler class $\text{Im } t \gg 0$ of $G_{k,n}$, the image Y of the total space X is given by $\mathcal{O}_{\mathbf{P}^{N-1}}(-n)$, restricted to the zero set of the system of quadrics, $\{Q_i = 0\} \forall i$. To describe the image $\hat{Y} = \varphi(\hat{X})$ of the singularity at small t , we consider the cone C over the set $\{Q_i = 0\} \subset \mathbf{P}^{N-1}$. Then C is the universal cover of \hat{Y} , which itself is obtained by dividing C by the \mathbf{Z}_n action $y_{\{r\}} \rightarrow \omega y_{\{r\}}$ with $\omega^n = 1$.

For example, for $k = 1$, $G_{1,n} = \mathbf{P}^{n-1}$, $x_\alpha = x_\alpha^1$ are the n homogeneous coordinates and the Plücker embedding is the identity map $\varphi : x_\alpha \rightarrow x_\alpha$. The singularity \hat{Y} is the cone over \mathbf{P}^{n-1} , divided by $x_\alpha \rightarrow \omega x_\alpha$ with $\omega^n = 1$. This is the same as $\mathbf{C}^n/\mathbf{Z}_n$ which may also be described as the cone over the n -th Veronese embedding of \mathbf{P}^{n-1} . For $k = 2$, the embedding φ maps the x_α^i to the homogeneous coordinates $y_{\alpha\beta} = \epsilon_{ij} x_\alpha^i x_\beta^j$ of \mathbf{P}^{N-1} with $N = n(n-1)/2$. There is one special case where the rank of the quadric system is maximal and $\varphi(G)$ is a complete intersection hypersurface, namely $G_{2,4}$. In this case the image of under the embedding φ is described by the zero locus of the single quadric $Q : y_{12}y_{34} + y_{13}y_{42} + y_{14}y_{23} = 0$ in \mathbf{P}^5 . The image of the total space $\mathcal{O}_G(-4)$ is the cone over Q divided by the \mathbf{Z}_4 that acts as $y_{\alpha\beta} \rightarrow iy_{\alpha\beta}$ on six coordinates of \mathbf{C}^6 .

Appendix B. The mirror model for $G_{2,5}$

As observed in [6], the McKay bases $\{S^a\}$ and $\{R_a\}$, the orthogonality relation (3.11) and the mutation $\underline{P} : \{R_a^\infty\} \rightarrow \{S_\infty^a\}$ have a very transparent interpretation in the W -plane of the mirror model. We restrict to the discussion of the mirror of the compact divisor $G_{k,n}$ in the following; the superpotential for the non-compact space obtained as in [30,5] has the same critical points, which is consistent with the fact that the non-compact direction does affect the morphisms between the ground states, but not the ground states themselves.

Concretely, the mirror of the sigma model on $G_{2,5}$ is (supposedly [27,30,5]) described by the Toda potential [31]

$$\begin{aligned} W = & X_1 + X_1^{-1}(X_2 + X_3) + X_2^{-1}X_6 + (X_2^{-1} + X_3^{-1})X_4 \\ & + (X_6^{-1} + X_4^{-1})X_5 + X_5^{-1} , \end{aligned} \tag{B.1}$$

where X_i are coordinates on $(\mathbf{C}^*)^6$, i.e. we may write $X_i = e^{-Y_i}$. The critical points of this potential in the W -plane reproduce precisely the nodes of the quiver diagram in Fig.1; in particular the distances in the diagram have now a meaning as the 2d masses of solitons that connect the critical points [32]. Special Lagrangian cycles correspond to straight lines in the W -plane [5].

We have indicated in Fig.2 the D-branes²⁴ on those SL cycles that are mirror to the sheaves $\{R_a^\infty\}$ (solid lines) and $\{S_\infty^a\}$ (dashed lines), respectively. The latter may be obtained a monodromy that pulls the solid paths through the critical points such as to obtain the dashed paths. This monodromy is the image under the mirror transformation of the mutation \underline{P} (4.6). In the small volume limit the critical points move to the origin and the compact SL cycles corresponding to $\{S_\infty^a\}$ collapse. Moreover the orthogonality relation (3.11) reflects the presence of the massless open strings sitting at the nodes, as indicated in the figure.

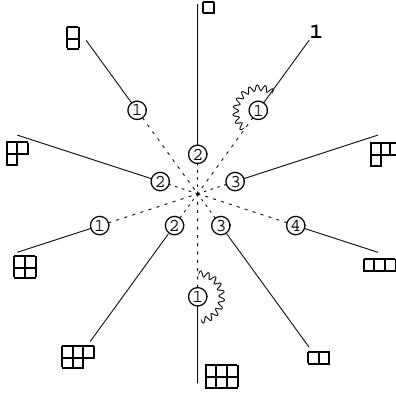


Fig.2: Special Lagrangian cycles in the LG mirror of $G_{2,5}$ in the W -plane of (B.1). The numbers give the fractional brane content of the $D0$ brane.

We have also indicated the fractional brane content of the $D0$ brane which according to the formula (4.11) is given by the ranks of the bundles R_a . The multiplicities of the fractional branes are quite large, and one may wonder whether there is a more canonical basis of generators $\{\tilde{R}_a\}$ and $\{\tilde{S}^a\}$ for which the multiplicities are smaller. An exceptional collection that leads to a minimal representation of the $D0$ brane may be obtained as follows. We have observed that there is a mutation of the $\{R_a\}$ that leads to a different exceptional collection $\{\tilde{R}_a\}$ with elements

$$\tilde{R}_a = \begin{cases} \det(E^*)^{\otimes (a-1)} & a \text{ odd,} \\ \det(E^*)^{\otimes (a-2)} \otimes E^* & a \text{ even.} \end{cases}$$

On general grounds [24] the sheaves \tilde{R}_a provide again free generators for $\mathcal{D}^b(X)$. The canonical ordering induced by the mutation is the series with an increasing number of boxes. The elements \tilde{R}_a are associated to the truncated modules $(\cdot, \begin{smallmatrix} \square \\ \square \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}, \begin{smallmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{smallmatrix})$ and $(\begin{smallmatrix} \square \\ \square \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}, \begin{smallmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{smallmatrix}, \begin{smallmatrix} \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \end{smallmatrix})$ of the coordinate algebra $\oplus \Sigma^{(i,i)} E^*$; this is similar as in

²⁴ We use here a labeling in terms of flipped Young tableaux, which directly give the ranks of the $U(2)$ bundles.

the algebraic description of $\mathcal{D}^b(G)$ in [9]. Curiously enough, the exchange of Young tableaux generated by this mutation corresponds precisely to identifications in the representation ring of $U(2)$ that project to the $U(2)$ fusion ring. These are of the form [18] $V_i U^j = V_{n-2-i} U^{j+n}$, with U the generator of the $U(1)$ and V_i elements of the $SU(2)$ fusion algebra at level $n = 5$, associated with the totally symmetric representations. Specifically, at small radius, the basis $\{\tilde{R}_a\}$ may be obtained by exchanging the ring elements that generate $\{R_a\}$ according to the identifications $\square \rightarrow \square\square\square$, $\square\square \rightarrow \square\square\square$, $\square\square\square \rightarrow \square\square\square\square$ of the $U(2)$ fusion algebra.

In terms of the exceptional collection $\{\tilde{S}^a\}$ defined in K-theory by $\tilde{S}_\infty^{a*} = (\chi(\tilde{R}_a^\infty)^{-1})^{ab} \tilde{R}_b^\infty$ and in the derived category by the sequences (4.6), the D0 brane class takes the following minimal form:

$$[D0] = 1 \times \sum_{a=1}^5 \tilde{S}^{2a-1} + 2 \times \sum_{a=1}^5 \tilde{S}^{2a} .$$

This corresponds to a quiver gauge group $U(1)^5 \times U(2)^5$; in the language of Fig.2, all the outer dots now carry ① while the inner dots carry ②. The change of the total number of component branes due to the mutation can be understood in terms of brane creation/annihilation processes induced by the braiding of the critical points of W [5].

Another feature of the new basis is that the intersection form on X becomes manifestly \mathbf{Z}_5 symmetric. These considerations may be important for a construction of the world-volume quiver theories as a quotient as in [25], which starts from r_a^2 D-branes for each node.

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